Math 206A Lecture 16 Notes

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November 2, 2018

1 Cauchy's Arm Lemma

1.1 The arm lemma

Lemma 1.1. *Let* $Q = [x_1, \dots, x_n], Q' = [x_1', \dots, x_n']$ *be two noncongruent convex polygons with equal corresponding lengths* $|x_i - x_{i+1}| = |x_1' - x_{i+1}'|$ *for* $i = 1, \dots, n \pmod n$*. Then ere exist at least 4 sign changes in* $\delta_i = \angle x_{i-1}x_ix_{i+1} - \angle x_{i-1}'x_i'x_{i+1}'$.

Example 1.1. Suppose $n = 4$, let Q be a square, and let Q' be a rhombus with angles α and $\pi - \alpha$. Then $\overline{\delta} = \{+, -, +, -\}$. The idea is that the diagonal length increases, which is impossible. Let's make this more rigorous.

Proof. Proceed by contradiction. Then $\overline{\delta} = \{+, +, \dots, +, -, -, \dots, -\}$. Then on one side of the polygon, the angles are increasing. □

Lemma 1.2 (arm lemma)**.** *Let* $P = [y_1, \dots, y_k]$, $P' = [y_1', \dots, y_k']$ *be convex polygons with* $|y_i - y_{i+1}| = |y_i' - y_{i+1}'|$ *for* $i = 1, \dots, k-1$*. If for* $i = 1, \dots, k-2$

$$\angle y_i y_{i+1} y_{i+2} \leq \angle y_i' y_{i+1}' y_{i+2}',$$

then $|y_1 - y_k| \leq |y_1' - y_k'|$.

1.2 Cauchy and Zaremba's proofs

Cauchy proved this in 1813 but incorrectly.[1] Let's go though Cauchy's proof.

Proof. Proceed by induction. When $n = 3$, we use the law of cosines. For the inductive step, increase all the angles except 1. Then, applying the law of cosines to the triangle formed by the triangle $x_1 x_n x_{n+1}$, we get that the length $x_1 - x_{n+1}$ increases. □

Where does this proof fail? It does not use convexity, and this theorem is not true for nonconvex polygons. There are cases where the inductive step does not work.

[1] He was about 19 at the time. Legendre gave him this as a project.

Proof. This is a proof by Zaremba.[2] In a case where Cauchy's proof doesn't work, first, increase the angle $\angle x_3 x_2 x_1$ until x_n lies on the segment connecting x_1 and x_{n-1} This is as far as we can expand the angle without losing convexity. Let the x_1' be the new point where x_1 is at. By the inductive hypothesis, the polygon $[x_1', \ldots, x_{n-1}]$ has the desired property, that is the line x_1' to x_{n-1} has gotten bigger (compared to $x_1 x_n$). So if you append a triangle onto this side to get a polygon with 1 more vertex, called x_n', the length of the segment connecting x_1 and x_n is smaller than the length of the segment connecting x_1' and x_n'. $\square$

[2]Zaremba and Schonberg corresponded, coming up with iterative constructions for this proof. Eventually, Zaremba came up with this proof. They published all three of their proofs. Basically, they jsut published their correspondence. But everyone only cares about the last proof.